# ON A CERTAIN CASE OF THE CONSTRUCTION OF PERIODIC SOLUTIONS OF QUASI-LINEAR SYSTEMS

**(OB ODNOM SLUCHAE POSTROENIIA PERIODICHESKIKH RESHENII KVAZILINEINYKH SISTEM)**

PMM Vol.28, № 5, 1964, pp.941-942

Iu.A.ARKHANGEL'SKII

(Moscow)

(Received April 14, 1964)

As is well known, the construction of periodic solutions of quasi-linear systems is based on the solution of the so-called equations of periodicity, which result from the periodicity conditions.

When the equations for the basic amplitudes, i.e. the periodicity equations with the small parameter set equal to zero, have a nonzero Jacobian, the equations for the basic amplitudes have the simple roots. This case has been studied rather fully by Malkin [1]. The periodic solutions so obtained can be expanded in integer powers of the small parameter.

When the Jacobian of the equations for the basic amplitudes vanishes, these equations have multiple roots. This case, with certain side conditions, has been studied by Proskuriakov and Plotnikova. In particular, they have treated non-self-contained systems with one degree of freedom in [2 and 3]. The periodic solutions obtained in this case can be expanded in either integer or fractional powers of the small parameter.

In these two cases, it is assumed that the considered values of the amplitudes of the generating solution are completely determined by the equations for the basic amplitudes.

However, for non-self-contained systems and self-contained systems with several degrees of freedom, cases can occur where for some values of the amplitude the generating solution of the system of equations for the basic amplitudes degenerates. This can mean (when the considered quasi-linear system has no first integral [4]) that the given form of the periodicity equations is not suitable for the determination of the corresponding amplitudes of the generating solution. In this case, the sought initial values of the periodic solution can be decomposed into two groups, to each of which corresponds a particular form of the periodicity equations.

1. Let us consider, for example, a quasi-linear non-self-contained system with one degree of freedom for which, following Proskuriakov [2], the periodicity equations, after cancelling the small parameter $\mu$, are represented in the form

$$\sum_{n=0}^{\infty} M_{jn}(A, B)\mu^n = 0, \qquad A = A_0 + \beta_1,\ B = B_0 + \beta_2 \qquad (j = 1, 2) \tag{1}$$

where $M_{jn}(A, B)$ are polinomials in $A$ and $B$. We will assume that Equations (1) take the form

$$f_1^{(1)}\varphi_0^{(1)} + \mu f_1^{(1)}\varphi_1^{(1)} + \cdots + \mu^{k-1} f_1^{(1)}\varphi_{k-1}^{(1)} + \mu^k f_1^* + \cdots = 0$$
$$f_2^{(1)}\varphi_0^{(1)} + \mu f_2^{(1)}\varphi_1^{(1)} + \cdots + \mu^{k-1} f_2^{(1)}\varphi_{k-1}^{(1)} + \mu^k f_2^* + \cdots = 0 \qquad (k > 0) \qquad (2)$$
$$\psi_0^{(1)} = f_2^{(1)} f_1^* - f_1^{(1)} f_2^* \not\equiv 0$$

Here, and in the sequel, $\varphi$ and $f$ are polinomials in $A$ and $B$, the $f_0^{(i)}$, $f_1^{(i)}$, $f_2^{(i)}$ have no common factor for each $t$, and the curves

$$\varphi_0^{(i)}(A,\ B) = 0$$

do not have isolated singular points. Dividing the equations of basic amplitudes in two groups

$$f_1^{(1)}(A_0, B_0) = 0, \quad f_2^{(1)}(A_0, B_0) = 0; \qquad \varphi_0^{(1)}(A_0, B_0) = 0 \qquad (3)$$

from the first group of equations we find the real roots $A_0^{(1)}$ and $B_0^{(1)}$ (if they exist), and from Equation (2) we find the corresponding quantities $\beta_1^{(1)}$ and $\beta_2^{(1)}$.

For determination of the remaining amplitudes of the generating equations we proceed as follows. Multiply the first equation of system (2) by $f_2^{(1)}$, and substract from it the second equation multiplied by $f_1^{(1)}$. Then, after cancelling $\mu^k$, we obtain an equation which, together with each of Equations (2), for example with the first, will be a new form of the periodicity equations

$$\psi_1^{(1)} + \mu(\ldots) = 0, \quad \psi_0^{(1)} + \mu(\ldots) = 0, \qquad \psi_j^{(1)} = f_j^{(1)}\varphi_0^{(1)} \qquad (j = 1,2) \qquad (4)$$

If the quantities $\psi_1^{(1)}$ and $\psi_0^{(1)}$ are represented in the form

$$\psi_1^{(1)} = f_1^{(2)}\varphi_0^{(2)}, \quad \psi_0^{(1)} = f_0^{(2)}\varphi_0^{(2)} \qquad (5)$$

then the periodicity equations (4) are analogous to Equations (2), and the entire process along the chosen direction is repeated until (as is assumed) at least one of the folloing inequalities is satisfied

$$\psi_1^{(l)} \neq f_1^{(l+1)}\varphi_0^{(l+1)}, \qquad \psi_0^{(l)} \neq f_0^{(l+1)}\varphi_0^{(l+1)} \qquad (l \geqslant 1)$$

In this case, from the equations for the basic amplitudes

$$\psi_1^{(l)}(A_0, B_0) = 0, \qquad \psi_0^{(l)}(A_0, B_0) = 0$$

we find the remaining amplitudes $A_0^{(l)}$, $B_0^{(l)}$ of the generating equations (if they exist), and from the periodicity equations

$$\psi_1^{(l)} + \mu(\ldots) = 0, \qquad \psi_0^{(l)} + \mu(\ldots) = 0 \qquad (6)$$

we find the corresponding quantities $\beta_1^{(l)}, \beta_2^{(l)}$. Then the process is repeated in another direction, etc. Thus, all initial values $A$, $B$ of the sought periodic solution are divided into groups to each of which corresponds a particular form of the periodicity equations. Similar reasoning can be used for self-contained and non-self-contained systems with several degrees of freedom.

2. As an example we will consider the Duffing problem in a quasi-linear formulation [1 and 2]. For the oscillation equation

$$\frac{d^2x}{dt^2} + x = \mu(\alpha x - \beta x^3) + \mu^2\gamma\cos t \qquad (\alpha\beta > 0),\ \gamma \neq 0) \qquad (7)$$

the generating solution will be $x_0 = A_0\cos t + B_0\sin t$, and the periodicity equations are

$$B\,[\alpha - {}^3/_4\beta(A^2 + B^2)] - \mu\,{}^3/_{32}\beta B\,[\alpha(3A^2 - B^2) - {}^1/_2\beta(7A^4 - 5B^4 + 8A^2B^2)] + \mu^2(\ldots) = 0 \qquad (8)$$

$$A\,[\alpha - {}^3/_4\beta(A^2 + B^2)] - \mu\,\{{}^1/_{32}\beta A\,[\alpha(A^2 - 3B^2) - {}^3/_2\beta(A^4 - 5B^4 + 4A^2B^2)] - \gamma\} + \mu^2(\ldots) = 0$$

The equations for the basic amplitudes are split up into the two groups

$$A_0 = 0, \quad B_0 = 0; \qquad \alpha - {}^3/_4\beta\,(A_0^2 + B_0^2) = 0$$

and the quantities $\beta_1^{(1)}$ and $\beta_2^{(1)}$, which correspond to vanishing amplitudes of the generating solution, are found from system (8). For the determination of the nonvanishing amplitudes of the generating solution we will transform system (8) into the form

$$f\,[\alpha - {}^3/_4\beta\,(A^2 + B^2)] + \mu\,(\ldots) = 0 \qquad (f = A, B)$$
$$B\,\{\gamma + {}^1/_{32}\beta A^3\,[8\alpha - 3\beta\,(3A^2 + 2B^2)]\} + \mu\,(\ldots) = 0 \qquad (9)$$

and obtain the equations of the basic amplitudes when it is assumed that $B_0 \neq 0$

$$A_0^2 + B_0^2 = {}^4/_3\alpha\beta^{-1}, \quad \gamma - {}^3/_{32}\beta^2 A_0^5 = 0$$

Hence, under the condition that $|\alpha|^5 > 27|\beta|\gamma^2$, we have

$$A_0 = 2\left(\frac{\gamma}{3\beta^2}\right)^{1/5}, \qquad B_0 = \pm\left[\frac{4\alpha}{3\beta} - 4\left(\frac{\gamma^2}{9\beta^4}\right)^{1/5}\right]^{1/2}$$

and $\beta_1^{(2)}$ and $\beta_2^{(2)}$, corresponding to these values of the amplitudes can be found from Equations (9).

BIBLIOGRAPHY

1. Malkin, I.G., Nekotorye zadachi teorii nelineinykh kolebanii (Some Problems of the Theory of Nonlinear Oscillations). Gostekhizdat, 1956.

2. Proskuriakov, A.P., Kolebaniia kvazilineinykh neavtonomnykh sistem s odnoi stepen'iu svobody vblizi rezonansa (Oscillations of quasi-linear non-self-contained systems with one degree of freedom near resonance). *PMM* Vol.23, № 5, 1959.

3. Plotnikova, G.V., O postroenii periodicheskikh reshenii neavtonomnoi sistemy s odnoi stepen'iu svobody vblizi rezonansa v sluchae dvukratnykh kornei uravnenii osnovnykh amplitud (On the construction of periodic solutions of a nonautonomous quasi-linear system with one degree of freedom near resonance in the case of double roots of the equation of fundamentsl amplitudes). *PMM* Vol.26, № 4, 1962.

4. Arkhangel'skii, Iu.A., Periodicheskie resheniia kvazilineinykh avtonomnykh sistem, obladaiushchikh pervymi integralami (Periodic solutions of qiasi-linear autonomous systems which have first integrals). *PMM* Vol.27, № 2, 1963.

Translated by D.B.McV.